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Free products of pseudocomplemented semilattices – revisited

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To Tibor Katriňák on the occasion of his 70th birthday

Abstract. We analyze the structure of free products of pseudocomplemented semilattices in terms of their skeletons and Glivenko classes by giving a rather explicit construction, complementing the description given by Katriňák and Heleyová in [6].

1. Introduction

Free products of pseudocomplemented semilattices (which, for brevity, we will refer to as p-semilattices) have been described by Katriňák and Heleyová in [6], giving an ingenious but intricate construction. Recently, the present authors had to construct explicitly a couple of very specific free products of p-semilattices in their pursuit of establishing relative (to Boolean algebras) universality of the category of all p-semilattices (see [1]). It became apparent that an approach to free products focussing on the structure of their Glivenko classes should work quite generally, generalizing the construction of free p-semilattices given by the second-named author in [8] and [9]. The purpose of this note is to provide a description of free products of p-semilattices which, we believe, is rather transparent. Moreover, it reduces directly to the the description of free p-semilattices cited above when forming free products of copies of the 1-generated free p-semilattice (the pentagon N_5). Terminology and notation is standard; for unexplained notions specific to p-semilattices, the reader is referred to [3, Chapter 3].

2. Set-up

Let $\mathbf{S}_i = (S_i \mid \wedge, ^*, 0, 1)$ for $i \in I$ be p-semilattices with associated order $\leq$. If necessary, operations, constants and order will be subscripted with i; **PCS** denotes the class of all p-semilattices. Define the *skeleton* of S_i to be $Sk(S_i) = \{\xi \in S_i \mid \xi = \xi^{**}\} = \{\xi \in S_i \mid \xi = \eta^* \text{ for some } \eta \in S_i\}$.

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Defining $\xi \veebar \eta = (\xi^* \wedge \eta^*)^*$ for $\xi, \eta \in Sk(S_i)$, we obtain a Boolean algebra $Sk(\mathbf{S}_i) = (Sk(S_i) \mid \wedge, \veebar, {}^*, 0, 1)$. Since the $(\wedge, {}^*, 0, 1)$-reduct of any Boolean algebra is a p-semilattice, $Sk(\mathbf{S}_i)$ becomes a sub-p-semilattice of $\mathbf{S}_i$ by dropping $\veebar$ from its type. Elements of any skeleton will also be called *Boolean*. Denote the class of all Boolean algebras by $\mathbf{BA}$. For notational convenience, we will use lower case Roman letters to denote Boolean elements whenever feasible, while lower case Greek letters will stand for any elements in p-semilattices (Boolean or not).

Let $\xi \in S_i$. We write $\Gamma_i(\xi)$ for $\{\eta \in S_i;\ \eta^{**} = \xi^{**}\}$, the *Glivenko class* of ξ in S_i. Each Glivenko class contains a unique Boolean element $a_i \in Sk(S_i)$, its greatest element under the induced order, and $\Gamma(a_i) = \{\xi_i \in S_i \mid \xi_i^{**} = a_i\}$. Moreover, the *Glivenko map* $g_i\colon \mathbf{S}_i \to Sk(\mathbf{S}_i)$ sending $\xi_i \in S_i$ to $\xi_i^{**} \in Sk(S_i)$ is a surjective $\mathbf{PCS}$-homomorphism. Note also that any $\mathbf{PCS}$-homomorphism between between Boolean algebras is a $\mathbf{BA}$-homomorphism, making $\mathbf{BA}$ a full subcategory of $\mathbf{PCS}$. The following simple fact is crucial for our purposes:

Fact 2.1. Any $\mathbf{PCS}$-homomorphism f from a p-semilattice $\mathbf{S}$ into any Boolean algebra $\mathbf{B}$ factors through the Glivenko map by $f = f{\restriction}Sk(S) \circ g$.

Proof. Assume $f\colon \mathbf{S} \to \mathbf{B}$ is a $\mathbf{PCS}$-homomorphism. Then for $\xi \in S$ one has $f(\xi) = f(\xi)^{**}$ as $\mathbf{B}$ is Boolean, and thus $f(\xi) = f(\xi^{**}) = f(g(\xi))$. It follows that $f = f{\restriction}Sk(S) \circ g$. □

For a family of p-semilattices $\mathbf{S}_i$ $(i \in I)$, a $\mathbf{PCS}$-free product consists of a p-semilattice $\mathbf{S}$ together with a family $\{e_i\colon \mathbf{S}_i \to \mathbf{S}\}$, for $i \in I$, of $\mathbf{PCS}$-embeddings, subject to the following two conditions: For any p-semilattice $\mathbf{T}$ and any family of $\mathbf{PCS}$-homomorphisms $t_i\colon \mathbf{S}_i \to \mathbf{T}$ $(i \in I)$, there exists a homomorphism $t\colon \mathbf{S} \to \mathbf{T}$ such that $t_i = t \circ e_i$ for all $i \in I$ and $\mathbf{S}$ is generated by $\bigcup_{i\in I} e_i(S_i)$. It follows that $\mathbf{S}$ is unique up to isomorphism whenever it exists, and will be denoted by $\coprod_{\mathbf{PCS}}\{\mathbf{S}_i \mid i \in I\}$ in the sequel.

The $\mathbf{BA}$-free product of any family $\{\mathbf{B}_i \mid i \in I\}$ of Boolean algebras, denoted by $\coprod_{\mathbf{BA}}\{\mathbf{B}_i \mid i \in I\}$, is defined similarly, using $\mathbf{BA}$-homomorphisms.

The *existence* of $\mathbf{PCS}$-free products may be deduced from very general theorems (see, e.g., [4, Section 29]), but our primary aim here is to provide a transparent *explicit construction*. So we only dispose of a trivial case at this point: If one of the p-semilattices $\mathbf{S}_i$ is the one-element algebra $\mathbf{1}$, then $\coprod_{\mathbf{PCS}}\{\mathbf{S}_i;\ i \in I\}$ exists if and only if $\mathbf{S}_i \cong \mathbf{1}$ for *all* $i \in I$, and then obviously $\coprod_{\mathbf{PCS}}\{\mathbf{S}_i \mid i \in I\} \cong \mathbf{1}$. This readily follows from the fact that there is no $\mathbf{PCS}$-embedding $e\colon \mathbf{1} \to \mathbf{S}$ unless $\mathbf{S} \cong \mathbf{1}$. So we assume henceforth that $\mathbf{S}_i \ncong \mathbf{1}$ for all $i \in I$ and, w.l.o.g. as we shall see, that $S_i \cap S_j = \{0, 1\}$ for $i \neq j$.

The following result is Proposition 1 of [6]. To make the paper reasonably self-contained, we include a proof adapted to our notational conventions.

Proposition 2.2. (Katriňák and Heleyová [6]) *If* $\mathbf{S} \cong \coprod_{\mathbf{PCS}}\{\mathbf{S}_i \mid i \in I\}$, *then* $Sk(\mathbf{S}) \cong \coprod_{\mathbf{BA}}\{Sk(\mathbf{S}_i) \mid i \in I\}$.

Proof. Consider $\mathbf{C} \in \mathbf{BA}$ and a family $\{f_i\colon Sk(\mathbf{S}_i) \to \mathbf{C}\}$, for $i \in I$, of Boolean homomorphisms. With $g_i\colon \mathbf{S}_i \to Sk(\mathbf{S}_i)$ the Glivenko maps, we define $f_i' := f_i \circ\ g_i\colon \mathbf{S}_i \to \mathbf{C}$ for $i \in I$. Since $\mathbf{S} \cong \coprod_{\mathbf{PCS}}\{\mathbf{S}_i \mid i \in I\}$, there exists a $\mathbf{PCS}$-homomorphism $t\colon \mathbf{S} \to \mathbf{C}$ such that $f_i' = t \circ e_i$ for all $i \in I$. Now $t = t\restriction Sk(\mathbf{S}) \circ g$ by Fact 2.1, where g is the Glivenko map $g\colon \mathbf{S} \to Sk(\mathbf{S})$; hence $f_i \circ g_i = f_i' = t\restriction Sk(\mathbf{S}) \circ g \circ e_i$. Let $\xi \in Sk(\mathbf{S}_i)$. Then $g_i(\xi) = \xi$ and $g(e_i(\xi)) = e_i(\xi)$, so that $f_i = t\restriction Sk(\mathbf{S}) \circ e_i\restriction Sk(\mathbf{S}_i)$ for all $i \in I$. Both $t\restriction Sk(\mathbf{S})$ and $e_i\restriction Sk(\mathbf{S}_i)$ are $\mathbf{BA}$-homomorphisms as $\mathbf{PCS}$-homomorphisms between Boolean algebras; so $t\restriction Sk(\mathbf{S})$ is the map required for $Sk(\mathbf{S})$ to be a free product of the $Sk(\mathbf{S}_i)$.

Since $\mathbf{S} = [\bigcup_{i\in I} e_i(S_i)]$, where $[\bigcup_{i\in I} e_i(S_i)]$ denotes the subalgebra generated by $\bigcup_{i\in I} e_i(S_i)$, we have $Sk(\mathbf{S}) \subseteq [\bigcup_{i\in I} e_i(S_i)]$. An induction on the length of $\xi \in S$ shows that for any ξ such that $\xi = \xi^{**}$, we have $\xi \in [\bigcup_{i\in I} e_i(Sk(S_i))]$. To see this, consider ξ for which $\xi = \xi^{**}$. If $\xi = \alpha^*$, then either $\alpha = \beta^*$ or $\alpha = \beta \wedge \gamma$. If $\alpha = \beta^*$, then (since $\beta^{***} = \beta^*$ in any p-semilattice) $\xi = \alpha^* \in [\bigcup_{i\in I} e_i(Sk(S_i))]$. If $\alpha = \beta \wedge \gamma$, then $\alpha^* = (\beta\wedge\gamma)^* = (\beta\wedge\gamma)^{***} = (\beta^{**}\wedge\gamma^{**})^*$ (since $(\beta\wedge\gamma)^{**} = \beta^{**}\wedge\gamma^{**}$ in any p-semilattice) and, since the lengths of β^* and γ^* are less than that of α, again $\xi = \alpha^* \in [\bigcup_{i\in I} e_i(Sk(S_i))]$. If, alternatively, $\xi = \alpha \wedge \gamma$, then $\xi = \xi^{**} = (\alpha\wedge\beta)^{**} = \alpha^{**}\wedge\beta^{**}$ and, since the lengths of α^* and β^* are less than that of ξ, as before $\xi = \alpha\wedge\gamma \in [\bigcup_{i\in I} e_i(Sk(S_i))]$. That is, as required, $Sk(\mathbf{S}) \subseteq [\bigcup_{i\in I} e_i(Sk(S_i))]$. □

3. The Boolean case

BA-free products of Boolean algebras may be described in different ways, for a purely algebraic approach see, e.g., [7, Chapter 4]. In order to facilitate the introduction of so-called *upper covers* – see below – and for the simplicity of the resulting construction, we adopt in this section an approach based on Stone duality, see, e.g., [10, Section 13].

Let $\{\mathbf{B}_i \mid i \in I\}$ be any family of Boolean algebras, and $\mathbf{X}_i = (X_i;\tau_i)$ the *Stone space* associated to $\mathbf{B}_i$ for $i \in I$. That is, the topology τ_i is Hausdorff, compact and totally disconnected, and $\mathbf{B}_i$ can be recovered from $\mathbf{X}_i$ as the collection of all clopen subsets of $\mathbf{X}_i$ endowed with set theoretic operations of union, intersection and complement, and $\emptyset$, respectively X_i, as constants. Conversely, $\mathbf{X}_i$ may be constructed as the space of all ultrafilters of $\mathbf{B}_i$ endowed with the hull-kernel topology, but this is not needed here. Moreover, $\mathbf{X}_i$ and $\mathbf{B}_i$ determine each other up to homeomorphism, respectively (Boolean) isomorphism.

Let $\mathbf{X} = (X;\tau) := \prod_{i\in I}\mathbf{X}_i$ be the topological product of the Stone spaces of the Boolean algebras $\mathbf{B}_i$. It is well-known that $\mathbf{X}$ is the Stone space of $\mathbf{B} = \coprod_{\mathbf{BA}}\{\mathbf{B}_i \mid i \in I\}$ (up to homeomorphism). In particular, a subset $U \subseteq X$ is associated with an element of B if and only if U is a clopen subset of X if and only if $U = \emptyset$ or U may be written (non-uniquely) in so-called normal form as a finite union $U = U_1 \cup \cdots \cup U_n$ where, for $1 \le j \le n$, $U_j = \prod_{i\in I} U_{ji}$

such that $\emptyset \neq U_{ji} \subseteq X_i$ is clopen and $U_{ji} \neq X_i$ for at most finitely many $i \in I$. The required **BA**-embeddings $e_i \colon \mathbf{B}_i \to \mathbf{B}$ are provided for all $j \in I$ by $e_i(V_i) = \pi_i^{-1}(V_i) = V_i \times \prod_{j\in I}(X_i \mid j \neq i)$ for a clopen subset $V_i \subseteq X_i$, where π_i is the canonical projection of $\mathbf{X}$ onto $\mathbf{X}_i$. It follows that U in normal form is a finite union of finite intersections of nonempty sets of type $e_i(V_i)$. Obviously, for U_j as above, one has

$$U_j \subseteq e_i(V_i) \text{ for some clopen } V_i \subseteq X_i \text{ iff } V_i \supseteq U_{ji}. \tag{3.1}$$

Writing $\mathbf{B}_i'$ for the canonical copy $e_i(\mathbf{B}_i)$ of $\mathbf{B}_i$ within $\mathbf{B}$, we define maps $\mathsf{c}_i \colon \mathbf{B} \to \mathbf{B}_i'$, for all $i \in I$, by $\mathsf{c}_i(U) := \pi_i^{-1} \circ \pi_i(U)$ for any clopen subset $U \subseteq X$. It is immediate that c_i distributes over finite unions (but not necessarily over finite intersections), hence c_i is order-preserving. If U is given in canonical form $U = U_1 \cup \cdots \cup U_n$, then $\mathsf{c}_i(U) = \mathsf{c}_i(U_1) \cup \cdots \cup \mathsf{c}(U_n) = U_{1i} \cup \cdots \cup U_{ni}$; thus $\mathsf{c}_i(U)$ is the smallest member of $\mathbf{B}_i'$ containing U by (3.1). It also follows from (3.1) that for any clopen $\emptyset \neq U \subseteq X$, one has $\mathsf{c}_i(U) \neq X_i$ for at most finitely many $i \in I$.

Reversing the process, we consider a finite subset $I_{fin} = \{i_1, \ldots, i_m\} \subseteq I$ and clopen subsets $\emptyset \neq V_i \subsetneq X_i$ for $i \in I_{fin}$, looking for clopen subsets $U \subseteq X$ such that $\mathsf{c}_i(U) = V_i$ for $i \in I_{fin}$ and $\mathsf{c}_i(U) = X_i$ for $i \notin I_{fin}$. Writing U in normal form, this amounts to requiring $U_{1i} \cup \cdots \cup U_{ni} = V_i$ for $i \in I_{fin}$ and $U_{1i} \cup \cdots \cup U_{ni} = X_i$ for $i \notin I_{fin}$. It follows that there is a unique largest such U, given by $U_{max} := \prod_{i\in I}(W_i \mid W_i = V_i \text{ for } i \in I_{fin} \text{ and } W_i = X_i \text{ otherwise})$.

Leaving Stone duality here for good, we will identify $\mathbf{B}_i$ and $\mathbf{B}_i'$ henceforth and think of c_i as a map from $\mathbf{B}$ into $\mathbf{B}_i$. So for $b \in B$, $\mathsf{c}_i b$ is the smallest element $x \in B_i$ such that $x \geq b$. For notational convenience, we will mostly write $b^{(i)}$ for $\mathsf{c}_i b$, for all $b \in B$ and $i \in I$, and call $b^{(i)}$ the *upper i-cover* of b. By the above arguments, c_i is a join-homomorphism, and thus order-preserving, but not a meet-homomorphism in general. Moreover, the following is a direct consequence of (3.1).

Fact 3.1. For $0 \neq b \in B$, $\mathsf{c}_i b \neq 1_i$ for at most finitely many $i \in I$.

Note also that by replacing $\mathbf{B}_i$ with $\mathbf{B}_i'$ we identify the zeros 0_i, respectively the units 1_i, of the algebras $\mathbf{B}_i$ with the zero 0, respectively the unit 1, of $\mathbf{B}$ — which is the main reason for postulating that $S_i \cap S_j = \{0, 1\}$ for $i \neq j$ at the end of section 2. However, we keep the subscripts on 0 and 1 whenever it seems desirable to track the history of such an element.

Any $b \in B$ may be written (non-uniquely) in disjunctive normal form, that is, as $b = b_1 \vee \cdots \vee b_n$ where each b_j $(1 \leq j \leq n)$ has the form $b_j = \bigwedge_{i\in I_j} b_i^j$ for some finite subset $I_j \subseteq I$ with $b_i^j \in B_i \setminus \{0_i\}$ for $i \in I_j$. Equation (3.1) above takes the form

$$b_j \leq a_i \text{ for } a_i \in B_i \text{ iff } a_i = 1_i \text{ or } (i \in I_j \text{ and } a_i \geq b_i^j). \tag{3.2}$$

It follows that $\mathsf{c}_i b_j = b_j^{(i)} = b_i^j$ iff $i \in I_j$ and $\mathsf{c}_i b_j = 1_i$ iff $i \notin I_j$, and thus $\mathsf{c}_i b = \mathsf{c}_i b_1 \vee \cdots \vee \mathsf{c}_i b_n$ equals either $b_i^1 \vee \cdots \vee b_i^n$ (if $i \in I_1 \cap \cdots \cap I_n$) or 1_i (otherwise).

We have thus an algebraic version for the construction of elements with prescribed i-covers:

Fact 3.2. Let $I_{fin} = \{i_1, \ldots, i_m\} \subseteq I$ be finite and $0 \neq a_i \in B_i$ for $i \in I_{fin}$. Then there exists a unique largest $b \in B$ such that $b^{(i)} = a_i$ for $i \in I_{fin}$ and $b^{(i)} = 1_i$ for $i \notin I_{fin}$, given explicitly as $b_{max} = a_{i_1} \wedge \cdots \wedge a_{i_m}$. More generally, $b \in B$ satisfies $b^{(i)} = a_i$ for $i \in I_{fin}$ and $b^{(i)} = 1_i$ for $i \notin I_{fin}$ iff b has a disjunctive normal form $b = b_1 \vee \cdots \vee b_n$ such that $I_{fin} = I_1 \cap \cdots \cap I_n$ and $a_i = b_i^1 \vee \cdots \vee b_i^n$ for $i \in I_{fin}$.

4. The Construction

Our plan is as follows: Suppose for the moment that $\mathbf{S} = \coprod_{\mathbf{PCS}}\{\mathbf{S}_i \mid i \in I\}$ exists, and let $\mathbf{B} := Sk(\mathbf{S})$ as well as $\mathbf{B}_i := Sk(\mathbf{S}_i)$ $(i \in I)$ in this section. Then we know by Proposition 2.2 that $\mathbf{B} = \coprod_{\mathbf{BA}}\{\mathbf{B}_i \mid i \in I\}$; so we aim to describe the members of $\Gamma(b)$ for any $b \in B$. Since S is generated by $\bigcup_{i \in I} S_i$, these members should be finite meets (taken within $\mathbf{S}$) of type $b \wedge \xi_{i_1} \wedge \cdots \wedge \xi_{i_n}$ with $b \in B$ and $\xi_{i_k} \in S_{i_k}$. We need not be concerned with pseudocomplements here since $\eta^* \in B$ for any $\eta \in S$.

Looking at the simplest case, consider $b \wedge \xi \in \Gamma(b)$ with $b \in B$ and $\xi \in S_i$. Since $b \leq b^{(i)}$, we have $b \wedge \xi = b \wedge \xi'$ where $\xi' = b^{(i)} \wedge \xi \in S_i$. Now $\xi'^{**} \leq (b^{(i)})^{**} = b^{(i)}$; on the other hand, $\xi'^{**} \geq (b \wedge \xi')^{**} = b$ (with the last equality holding since $b \wedge \xi' = b \wedge \xi \in \Gamma(b)$). As $b^{(i)}$ is the upper i-cover of b, we conclude $\xi'^{**} = b^{(i)}$. It follows that any meet $b \wedge \xi \in \Gamma(b)$ with $\xi \in S_i$ may be written as $b \wedge \xi'$ with $\xi' \in \Gamma_i(b^{(i)})$. This motivates the following definition:

Definition 4.1. For $b \in B$, let $\Lambda(b)$ be the collection of all $\beta = (\beta_i) \in \prod_{i \in I} S_i$ such that $\beta_i \in \Gamma_i(b^{(i)})$ for all $i \in I$ but $\beta_i \neq b^{(i)}$ for at most finitely many $i \in I$. Put $P(b) := \{(b; \beta); \beta \in \Lambda(b)\}$, and let $P := \bigcup_{b \in B} P(b)$.

Remark 4.2. It may help to think of the members of P as coding up the hypothetical meets (in $\mathbf{S}$) of type $b \wedge \xi_{i_1} \wedge \cdots \wedge \xi_{i_n}$ with $\xi_{i_k} \in S_{i_k}$, as considered above (recall Fact 3.1). This will be made precise by Corollary 4.9.

Notation and nomenclature 4.3. Elements of P will be written as $\boldsymbol{\beta} = (b; \beta) = (b; (\beta_i))$, and analogously for other lower-case roman and corresponding greek letters. We call the Boolean element $b \in B$ the *head* and the $\beta_i \in \Gamma_i(b^{(i)})$) the *components* of $\boldsymbol{\beta}$; the latter are also referred to by $(\boldsymbol{\beta})_i := \beta_i$.

As $P \subseteq B \times \prod_{i \in I} S_i$, the product order on $B \times \prod_{i \in I} S_i$ induces an order $\leq_P$ on P. The main result of this paper (Theorem 4.10) is that $\leq_P$ induces the structure of a pseudocomplemented semilattice on $(P; \leq_P)$ which makes it a $\mathbf{PCS}$-free product of the $\mathbf{S}_i$.

Lemma 4.4. *$(P; \leq_P)$ is a meet-semilattice with least element $0_P := (0; (0_i))$ and greatest element $1_P := (1; (1_i))$. The meet of $\boldsymbol{\alpha} = (a; (\alpha_i))$, $\boldsymbol{\beta} = (b; (\beta_i))$ in $(P; \leq_P)$ is given by $\boldsymbol{\alpha} \wedge_P \boldsymbol{\beta} := (a \wedge b; (\alpha_i \wedge \beta_i \wedge (a \wedge b)^{(i)}))$.*

Proof. Consider $\boldsymbol{\gamma} = (c; (\gamma_i)) \in P$ such that $\boldsymbol{\gamma} \leq \boldsymbol{\alpha}, \boldsymbol{\beta}$. The definition of $\leq_P$ gives $c \leq a \wedge b$ and $\gamma_i \leq \alpha_i \wedge \beta_i$. Since $\boldsymbol{\gamma} \in P$, one also has $\gamma_i^{**} = c^{(i)}$, thus $\gamma_i \leq \gamma_i^{**} = c^{(i)} \leq (a \wedge b)^{(i)}$ and so $\gamma_i \leq \alpha_i \wedge \beta_i \wedge (a \wedge b)^{(i)}$. It remains to show that $(a \wedge b; (\alpha_i \wedge \beta_i \wedge (a \wedge b)^{(i)}))$ actually is in P. But $(\alpha_i \wedge \beta_i \wedge (a \wedge b)^{(i)})^{**} = \alpha_i^{**} \wedge \beta_i^{**} \wedge ((a \wedge b)^{(i)})^{**} = a^{(i)} \wedge b^{(i)} \wedge (a \wedge b)^{(i)} = (a \wedge b)^{(i)}$ as $(a \wedge b)^{(i)} \leq a^{(i)} \wedge b^{(i)}$ (since c_i is order-preserving). □

The subscripts $_P$ on $\leq$, $\wedge$, 0 and 1 may be dropped henceforth, since, hopefully, there is no danger of confusion.

Lemma 4.5. *Let $(P; \wedge, 0, 1)$ be the semilattice of Lemma 4.4. Then $(P; \wedge, 0, 1)$ is a pseudocomplemented semilattice, the pseudocomplement of $\boldsymbol{\alpha} = (a; (\alpha_i))$ being given by $\boldsymbol{\alpha}^{*_P} = (a^*; ((a^*)^{(i)}))$.*

Proof. Fix $\boldsymbol{\alpha} = (a; (\alpha_i))$ and suppose $\boldsymbol{\alpha} \wedge \boldsymbol{\gamma} = 0$ for some $\boldsymbol{\gamma} = (c; (\gamma_i))$. By Lemma 4.4 we have $a \wedge c = 0$, that is, $c \leq a^*$, and so $\gamma_i \leq c^{(i)} \leq (a^*)^{(i)}$. On the other hand, $(a; (\alpha_i)) \wedge (a^*, ((a^*)^{(i)})) = (a \wedge a^*; (\alpha_i \wedge (a^*)^{(i)} \wedge (a \wedge a^*)^{(i)})) = (0; (0_i))$ since $(a \wedge a^*)^{(i)} = 0^{(i)} = 0_i$. □

We write $\mathbf{P}$ for the p-semilattice $\mathbf{P} = (P; \wedge, ^*, 0, 1) = (P; \wedge_P, ^{*_P}, 0_P, 1_P)$.

Define a map $\iota \colon B \to P$ by $\iota(a) := (a; (a^{(i)}))$ for all $a \in B$.

Corollary 4.6. *The map ι is a (Boolean) isomorphism from $\mathbf{B}$ onto $Sk(\mathbf{P})$. Accordingly, we will also write $\mathbf{B}' := \iota(\mathbf{B})$ for the skeleton of $\mathbf{P}$.*

Proof. By Lemma 4.5, $Sk(\mathbf{P}) = \{\boldsymbol{\alpha}^* \mid \boldsymbol{\alpha} \in P\} = \{(a^*; ((a^{(i)})^*)) \mid a \in B\} = \{(a; (a^{(i)})) \mid a \in B\}$ since * acts bijectively on B, respectively B_i, so ι is bijective. Let $a, b \in B$, then

$$\begin{aligned} \iota(a) \wedge \iota(b) &= (a; (a^{(i)})) \wedge (b; (b^{(i)})) = (a \wedge b; (a^{(i)} \wedge b^{(i)} \wedge (a \wedge b)^{(i)})) \\ &= (a \wedge b; ((a \wedge b)^{(i)})) = \iota(a \wedge b), \end{aligned}$$

since $(a \wedge b)^{(i)} \leq a^{(i)} \wedge b^{(i)}$, and $(\iota(a))^* = (a; (a^{(i)}))^* = (a^*; ((a^*)^{(i)})) = \iota(a^*)$. Since Boolean join on $\mathbf{B}$, respectively $Sk(\mathbf{P})$, is defined in terms of $\wedge$ and *, it is also preserved. □

Define, for each $j \in I$, a map $\mathsf{e}_j \colon S_j \to P$ by $\mathsf{e}_j(\xi) := (\xi^{**}; (\xi_i))$ for $\xi \in S_j$, where $\xi_i = (\xi^{**})^{(i)}$ (if $i \neq j$) and $\xi_i = \xi$ (if $i = j$). Explicitly, $\mathsf{e}_j(\xi) = (0; (0_i))$ if $\xi = 0$, and $\mathsf{e}_j(\xi) = (\xi^{**}; (1, dots, 1, \xi, 1, \dots, 1))$ with ξ at the j-th position if $\xi \neq 0$.

Lemma 4.7. *e_j embeds $\mathbf{S}_j$ into $\mathbf{P}$ (as p-semilattices) for all $j \in I$, and $\mathsf{e}_j \restriction B_j$ embeds $\mathbf{B}_j$ into $\mathbf{B}' = Sk(\mathbf{P})$ (as Boolean algebras).*

Proof. Consider $\xi, \eta \in S_j$ and put $x := \xi^{**}$, $y := \eta^{**}$. The heads of $\mathsf{e}_j(\xi)$, respectively $\mathsf{e}_j(\eta)$, thus are x, respectively y, while their components are $x^{(i)}$, respectively $y^{(i)}$, for $i \neq j$ and ξ, respectively η, for $i = j$. This shows that e_j is one-to-one.

Similarly, $\mathsf{e}_j(\xi \wedge \eta)$ has head $(\xi \wedge \eta)^{**} = \xi^{**} \wedge \eta^{**} = x \wedge y$ and components $(x \wedge y)^{(i)}$ for $i \neq j$, respectively $\xi \wedge \eta$ for $i = j$. Compute $\mathsf{e}_j(\xi) \wedge \mathsf{e}_j(\eta)$ according to Lemma 4.4 and obtain $x \wedge y$ for its head, $x^{(i)} \wedge y^{(i)} \wedge (x \wedge y)^{(i)}$ for $i \neq j$, respectively $\xi \wedge \eta \wedge (x \wedge y)^{(j)}$ for $i = j$, for its components. If $i \neq j$, then $x^{(i)} \wedge y^{(i)} \wedge (x \wedge y)^{(i)} = (x \wedge y)^{(i)}$ since c_i is order-preserving, while $\xi \wedge \eta \wedge (x \wedge y)^{(j)} = \xi \wedge \eta$ since $\xi \wedge \eta \leq x \wedge y \leq (x \wedge y)^{(j)}$. Consequently, $\mathsf{e}_j(\xi \wedge \eta) = \mathsf{e}_j(\xi) \wedge \mathsf{e}_j(\eta)$.

It remains to show that $\mathsf{e}_j(\xi^*) = (\mathsf{e}_j(\xi))^*$. According to 4.5, $(\mathsf{e}_j(\xi))^*$ has head x^* and components $(x^*)^{(i)}$ for all $i \in I$. On the other hand, $\mathsf{e}_j(\xi^*)$ has head $\xi^{***} = x^*$, components $(x^*)^{(i)}$ (if $i \neq j$) and ξ^* (if $i = j$). But $(x^*)^{(j)} = (\xi^{***})^{(j)} = (\xi^*)^{(j)} = \xi^*$ since $\xi^* \in B_j$, and thus $\mathsf{e}_j(\xi^*) = (\mathsf{e}_i(\xi))^*$.

If $\xi \in B_i$, then $\xi = \xi^{**} = x$, and thus $\mathsf{e}_j(\xi) \in \mathbf{B}'$. As above, Boolean join is preserved automatically. $\square$

It follows that $\mathbf{S}'_j := \mathsf{e}_j(\mathbf{S}_j)$ is a subalgebra of $\mathbf{P}$ isomorphic to $\mathbf{S}_j$, and analogously, $\mathbf{B}'_j := \mathsf{e}_j(\mathbf{B}_j)$ a Boolean subalgebra of $\mathbf{B}'$ isomorphic to $\mathbf{B}_j$, for all $j \in I$.

Lemma 4.8. (i) *The union of all subalgebras $\mathbf{S}'_j$ $(j \in I)$ generates $\mathbf{P}$ as a p-semilattice.*

(ii) *The union of all subalgebras $\mathbf{B}'_j$ $(j \in I)$ generates $\mathbf{B}'$ as a Boolean algebra.*

Proof. We start with (ii). Consider $b \in B$, thus $b = p(b_{j_1}, \ldots, b_{j_n})$ where p is a Boolean polynomial and $b_{j_k} \in B_{j_k} \subseteq B$ for $1 \leq k \leq n$. Hence $\iota(b) = \iota(p(b_{j_1}, \ldots, b_{j_n})) = p(\iota(b_{j_1}), \ldots, \iota(b_{j_n}))$. But $\iota(b_{j_k}) = (b_{j_k}; (b_{j_k}^{(i)})) = \mathsf{e}_{j_k}(b_{j_k}) \in \mathbf{B}'_{j_k}$ for $1 \leq k \leq n$ and we are done.

For (i), consider $\boldsymbol{\beta} = (b; (\beta_i)) \in \mathbf{P}$. It follows from (ii) above that $\boldsymbol{\beta}^{**} = (b; (b^{(i)}))$ is in the subalgebra of $\mathbf{P}$ generated by $\bigcup_{j \in I} \mathbf{S}'_j$. Put $\{j_1, dots, j_n\} := \{j \in I \mid \beta_j \neq b^{(j)}\}$. Hence, for $1 \leq k \leq n$, we have $\mathsf{e}_{j_k}(\beta_{j_k}) = (\xi^{**}; (\xi_i))$ with $\xi^{**} = (\beta_{j_k})^{**} = b^{(j_k)} \geq b$, $\xi_i = 1$ for $i \neq j_k$ and $\xi_i = \beta_{j_k}$ for $i = j_k$. Then $\boldsymbol{\beta}^{**} \wedge \mathsf{e}_{j_1}(\beta_{j_1}) \wedge \cdots \wedge \mathsf{e}_{j_n}(\beta_{j_n}) = (b; (\eta_i))$ where $\eta_i = b^{(i)}$ for $i \notin \{j_1, \ldots, j_n\}$ (the i-th components of all $\mathsf{e}_{j_k}(\beta_{j_k})$ being 1) while $\eta_{j_k} = \beta_{j_k} \wedge b^{(j_k)} = \beta_{j_k}$ for $i = j_k$ with $1 \leq k \leq n$ (the i-th components of $\mathsf{e}_{j_{k'}}(\beta_{j_{k'}})$ being 1 for $k' \neq k$). But this means that $(b; (\eta_i)) = (b; (\beta_i)) = \boldsymbol{\beta}$, and $\boldsymbol{\beta}$ is in the subalgebra of $\mathbf{P}$ generated by $\bigcup_{j \in I} \mathbf{S}'_j$. $\square$

Corollary 4.9 (Normal forms)**.** *For all $\boldsymbol{\beta} = (b; (\beta_i)) \in \mathbf{P}$, we have $\boldsymbol{\beta} = \boldsymbol{\beta}^{**} \wedge \bigwedge_{i \in I} \mathsf{e}_i(\beta_i)$.*

Proof. In the proof of item (i) just above, we obtained

$$\boldsymbol{\beta} = \boldsymbol{\beta}^{**} \wedge \mathsf{e}_{j_1}(\beta_{j_1}) \wedge \cdots \wedge \mathsf{e}_{j_n}(\beta_{j_n}),$$

where $\{j_1, \ldots, j_n\} = \{j \in I \mid \beta_j \neq b^{(j)}\}$. Suppose $\beta_j = b^{(j)}$. Then $\mathsf{e}_j(\beta_j) = \mathsf{e}_j(b^{(j)})$, but

$$\mathsf{e}_j(b^{(j)}) = ((b^{(j)})^{**}; (1, \ldots, 1, b^{(j)}, 1, \ldots, 1)) = ((b^{(j)}); (1, \ldots, 1, b^{(j)}, 1, \ldots, 1))$$

and

$$((b^{(j)}); (1, \ldots, 1, b^{(j)}, 1, \ldots, 1)) \geq (b; (b^{(i)})) = \boldsymbol{\beta}^{**},$$

so these terms cancel out from $\bigwedge_{i \in I} \mathsf{e}_i(\beta_i)$ when forming the meet with $\boldsymbol{\beta}^{**}$. □

Theorem 4.10. **P** *is a* **PCS**-*free product of the p-semilattices* $\mathbf{S}_i$ *(*$i \in I$*).*

Proof. We will show that **P** is a **PCS**-free product of its subalgebras $\mathbf{S}'_i$ $(i \in I)$. To this end, let **T** be an arbitrary p-semilattice together with a family of p-semilattice homomorphisms $t'_i \colon \mathbf{S}'_i \to \mathbf{T}$ for $i \in T$. We will construct a p-semilattice homomorphism $t \colon \mathbf{P} \to \mathbf{T}$ extending t'_i for all $i \in I$.

Note that $t'_i \restriction \mathbf{B}'_i$ is a Boolean homomorphism from $\mathbf{B}'_i$ to $Sk(\mathbf{T})$, so there exists a Boolean homomorphism $t'_B \colon \mathbf{B}' \to Sk(\mathbf{T})$ extending $t'_i \restriction \mathbf{B}'_i$ for all $i \in I$ as $\mathbf{B}'$ is a **BA**-free product of its subalgebras $\mathbf{B}'_i$. Consider $\boldsymbol{\beta} \in P$ where $\boldsymbol{\beta} = (b; (\beta_i))$. By Corollary 4.9, we have $\boldsymbol{\beta} = \boldsymbol{\beta}^{**} \wedge \bigwedge_{i \in I} \mathsf{e}_i(\beta_i)$.

Define $t(\boldsymbol{\beta}) := t'_B(\boldsymbol{\beta}^{**}) \wedge \bigwedge_{i \in I} t'_i(\mathsf{e}_i(\beta_i))$. We have to show that t is compatible with $\wedge$ and *.

Starting with meet, consider $\boldsymbol{\alpha} = (a; (\alpha_i))$ and $\boldsymbol{\beta} = (b; (\beta_i))$ in **P**. We note first that

$$\begin{aligned}(\boldsymbol{\alpha} \wedge \boldsymbol{\beta})^{**} &= (a \wedge b; ((a \wedge b)^{(i)}) \leq ((a \wedge b)^{(i)}; (1, \ldots, 1, (a \wedge b)^{(i)}, 1, \ldots, 1)) \\ &= \mathsf{e}_i((a \wedge b)^{(i)}).\end{aligned}$$

Since $\mathsf{e}_i((a \wedge b)^{(i)}) \in \mathbf{B}'_i$, this implies $t'_i(\mathsf{e}_i((a \wedge b)^{(i)})) = t'_B(\mathsf{e}_i((a \wedge b)^{(i)}))$ and thus

$$t'_B((\boldsymbol{\alpha} \wedge \boldsymbol{\beta})^{**}) \leq t'_i(\mathsf{e}_i((a \wedge b)^{(i)})) \tag{4.1}$$

for all $i \in I$. Now compute

$$\begin{aligned}t(\boldsymbol{\alpha}) \wedge t(\boldsymbol{\beta}) &= t'_B(\boldsymbol{\alpha}^{**}) \wedge \bigwedge_{i \in I} t'_i(\mathsf{e}_i(\alpha_i)) \wedge t'_B(\boldsymbol{\beta}^{**}) \wedge \bigwedge_{i \in I} t'_i(\mathsf{e}_i(\beta_i)) \\ &= t'_B((\boldsymbol{\alpha} \wedge \boldsymbol{\beta})^{**}) \wedge \bigwedge_{i \in I} (t'_i(\mathsf{e}_i(\alpha_i)) \wedge t'_i(\mathsf{e}_i(\beta_i)))\end{aligned}$$

which, by (4.1),

$$\begin{aligned}&= t'_B((\boldsymbol{\alpha} \wedge \boldsymbol{\beta})^{**}) \wedge \bigwedge_{i \in I} (t'_i(\mathsf{e}_i(\alpha_i)) \wedge t'_i(\mathsf{e}_i(\beta_i)) \wedge t'_i(\mathsf{e}_i((a \wedge b)^{(i)}))) \\ &= t'_B((\boldsymbol{\alpha} \wedge \boldsymbol{\beta})^{**}) \wedge \bigwedge_{i \in I} t'_i(\mathsf{e}_i(\alpha_i \wedge \beta_i \wedge (a \wedge b)^{(i)})) \\ &= t(\boldsymbol{\alpha} \wedge \boldsymbol{\beta}),\end{aligned}$$

since the $\alpha_i \wedge \beta_i \wedge (a \wedge b)^{(i)})$ are the components of $\boldsymbol{\alpha} \wedge \boldsymbol{\beta}$.

For pseudocomplements, put $\boldsymbol{\alpha} = \boldsymbol{\beta}$ in (3) to obtain $t'_B(\boldsymbol{\alpha}^{**}) \leq t'_i(\mathsf{e}_i(a^{(i)}))$ and observe that $t'_i(\mathsf{e}_i(a^{(i)})) = t'_i(\mathsf{e}_i(\alpha_i^{**})) = (t'_i(\mathsf{e}_i(\alpha_i))^{**}$, thus

$$t'_B(\boldsymbol{\alpha}^{**}) \leq (t'_i(\mathsf{e}_i(\alpha_i))^{**} \tag{4.2}$$

for all $i \in I$. With that, we obtain

$$(t(\boldsymbol{\alpha}))^{**} = (t'_B(\boldsymbol{\alpha}^{**}))^{**} \wedge \bigwedge_{i \in I} (t'_i(\mathsf{e}_i(\alpha_i)))^{**} = (t'_B(\boldsymbol{\alpha}^{**}))^{**} = t'_B(\boldsymbol{\alpha}^{**})$$

by (4.2) (distributing ** over $\bigwedge$ is in order since the meet may be replaced by a finitary one, see the proof of Corollary 4.9). Forming pseudocomplements once more, we get $t(\boldsymbol{\alpha})^* = (t'_B(\boldsymbol{\alpha}^{**}))^* = t'_B(\boldsymbol{\alpha}^*)$.

If $\boldsymbol{\alpha} \in \mathbf{B}'$, putting $\boldsymbol{\alpha} = \boldsymbol{\beta}$ in (4.1) results in $t'_B(\boldsymbol{\alpha}) = t'_B(\boldsymbol{\alpha}^{**}) \leq t'_i(\mathsf{e}_i(a^{(i)})) = t'_B(\mathsf{e}_i(a^{(i)}))$, and thus $t(\boldsymbol{\alpha}) = t'_B(\boldsymbol{\alpha}^{**}) \wedge \bigwedge_{i \in I} t'_B(\mathsf{e}_i(a^{(i)})) = t'_B(\boldsymbol{\alpha})$. It follows that $t \restriction \mathbf{B}' = t'_B$. We conclude that $t(\boldsymbol{\alpha}^*) = t'_B(\boldsymbol{\alpha}^*)$, and since also $t(\boldsymbol{\alpha})^* = t'_B(\boldsymbol{\alpha}^*)$ as seen above, we obtain $t(\boldsymbol{\alpha})^* = t(\boldsymbol{\alpha}^*)$ as desired.

It remains to show that t extends t'_j for all $j \in I$. Consider $\mathsf{e}_j(\xi) \in \mathbf{S}_j$. Remembering that $(\boldsymbol{\alpha})_i$ stands for the i-th component of $\boldsymbol{\alpha}$, compute $t(\mathsf{e}_j(\xi)) = t'_B((\mathsf{e}_j(\xi)^{**})) \wedge \bigwedge_{i \in I} t'_i(\mathsf{e}_i((\mathsf{e}_j(\xi))_i))$. Now $t'_B((\mathsf{e}_j(\xi)^{**})) = t'_j((\mathsf{e}_j(\xi))^{**})$ because $(\mathsf{e}_j(\xi)^{**}) \in \mathbf{B}'_j$ and t'_B extends $t'_j \restriction \mathbf{B}'_j$. Since $(\mathsf{e}_j(\xi))_i = 1$ for $i \neq j$ and $(\mathsf{e}_j(\xi))_j = \xi$, we have $t'_i(\mathsf{e}_i((\mathsf{e}_j(\xi))_i)) = 1$ for $i \neq j$ and $t'_j(\mathsf{e}_j((\mathsf{e}_j(\xi))_j)) = t'_j(\mathsf{e}_j(\xi))$. Hence $t(\mathsf{e}_j(\xi)) = t'_B((\mathsf{e}_j(\xi)^{**})) \wedge \bigwedge_{i \in I} t'_i(\mathsf{e}_i((\mathsf{e}_j(\xi))_i))$ reduces to $t'_j((\mathsf{e}_j(\xi))^{**}) \wedge t'_j(\mathsf{e}_j(\xi))$ which equals $t'_j(\mathsf{e}_j(\xi))$, and we are done. □

5. Free p-semilattices

Free p-semilattices were first described in [5] and subsequently in Balbes [2] and, of particular interest here, [8] and [9], using different approaches. Another description results in [6] from specializing the free product construction given there. Here, we show that the construction obtained in Section 4 specializes to the one given in [8].

Let $\mathbf{F_{PCS}}(I)$ be the p-semilattice freely generated by a set $\{g_i \mid i \in I\}$ of free generators. It is immediate that $\mathbf{F_{PCS}}(\emptyset) \cong \mathbf{2}$, the 2-element p-semilattice, and $\mathbf{F_{PCS}}(\{i\}) \cong \mathbf{N_5}$, the p-semilattice reduct of the pentagon realized as $\{0, g_i, g_i^*, g_i^{**}, 1\}$. Hence, $\mathbf{F_{PCS}}(I) \cong \coprod_{\mathbf{PCS}} \{(\mathbf{N_5})_i \mid i \in I\}$, the free product of $|I|$ many copies of $\mathbf{N_5}$. By Proposition 2.2, we have

$$Sk(\mathbf{F_{PCS}}(I)) \cong \coprod_{\mathbf{BA}} \{(\mathbf{2}^2)_i \mid i \in I\} \cong \mathbf{F_{BA}}(I),$$

the Boolean algebra freely generated by $|I|$ many generators.

So let $\mathbf{B} = \mathbf{F_{BA}}(I)$ and select, w.l.o.g., $\{g_i^{**} \mid i \in I\}$ as its set of free generators. For any $0 \neq b \in B$ and $i \in I$, we have $b^{(i)} \in \{g_i^*, g_i^{**}, 1_i\}$, with $b^{(i)} \neq 1_i$ for at most finitely many $i \in I$. Conversely, let $I_{fin} \subseteq I$ be any finite subset of I and consider $(b_i)_{i \in I}$ with $b_i \in \{g_i^*, g_i^{**}\}$ for $i \in I_{fin}$ and $b_i = 1_i$ for $i \notin I_{fin}$. It is easy to describe the elements $b \in B$ satisfying $b^{(i)} = b_i$ for all $i \in I$: Obviously $0 \neq b$ and thus $0 < b^{(i)}$, hence $b \leq b^{(i)} \in \{g_i^*, g_i^{**}, 1_i\}$. It follows that b has the required property if and only if $b \leq g_i^*$, respectively $b \leq g_i^{**}$, for $i \in I_{fin}$ and $b \not\leq g_i^*$ as well as $b \not\leq g_i^{**}$ for $i \notin I_{fin}$. By Fact 3.2, b is of this type if and only if it has a disjunctive normal form where exactly the g_i^*, respectively the g_i^{**}, with $i \in I_{fin}$ occur as common meetands in *all* meet terms.

Passing to $\mathbf{F_{PCS}}(I)$, we analyze its structure by describing the Glivenko class $\Gamma(\boldsymbol{\beta}) \subseteq \mathbf{F_{PCS}}(I)$ for each $0 \neq \boldsymbol{\beta} \in Sk(\mathbf{F_{PCS}}(I)) = \mathbf{B}' \cong \iota(\mathbf{B})$. In this case, $\boldsymbol{\beta}$ has the form $\boldsymbol{\beta} = (b; (b^{(i)}))$ with $b \in B$ and such that $b^{(i)}) \in \{g_i^*, g_i^{**}\}$ for some finite subset $I_{fin} \subseteq I$ and $b^{(i)}) = 1_i$ for $i \notin I_{fin}$. Consider $\boldsymbol{\alpha} = (a; (\alpha_i)) \in \mathbf{F_{PCS}}(I)$ such that $\boldsymbol{\alpha}^{**} = \boldsymbol{\beta}$. It follows that $\boldsymbol{\alpha}$ may be obtained from $\boldsymbol{\beta}$ by replacing some or all of the occurrences of g_i^{**} in $\boldsymbol{\beta}$ by g_i. Let $\mu(\boldsymbol{\beta})$ be the number of g_i^{**}'s occurring in $\boldsymbol{\beta}$. Then $\Gamma(\boldsymbol{\beta})$ is isomorphic, in its induced order, to $\mathbf{2}^{\mu(\beta)}$, the Boolean lattice with $\mu(\boldsymbol{\beta})$ atoms. Summing up, we have

Theorem 5.1. *As an ordered set,* $\mathbf{F_{PCS}}(I) \setminus \{0\}$ *is isomorphic to*

$$\{\boldsymbol{\beta} = (b; (\beta_i)) \mid 0 \neq b \in \mathbf{F_{BA}}(I),\, b^{(i)} = \beta_i^{**} \in \{g_i^*, g_i^{**}\}$$

for at most finitely many $i \in I$*, and* $\beta_i = 1_i$ *elsewhere, under the componentwise order. For such* $\boldsymbol{\beta}$*, one has* $\Gamma(\boldsymbol{\beta}) = \Gamma(\boldsymbol{\beta}^{**}) \cong \mathbf{2}^{\mu(\boldsymbol{\beta}^{**})}$. *Explicitly,* $\Gamma(\boldsymbol{\beta})$ *is the set of all* $\iota(b) \wedge \boldsymbol{\xi}$ *where* $\boldsymbol{\xi}$ *is a finite meet of elements* $\mathsf{e}_i(g_i)$*, respectively* $\mathsf{e}_i(g_i^{**})$*, (if* $b^{(i)} = g_i^{**}$*) and* $\mathsf{e}_i(g_i^*)$ *(if* $b^{(i)} = g_i^*$*). Conversely, given* (β_i) *with* $\beta_i^{**} \in \{g_i^*, g_i^{**}\}$ *for at most finitely many* $i \in I$ *and* $\beta_i = 1_i$ *elsewhere, then* $\Gamma((b; (\beta_i)))$ *has this form exactly if* $b \in \mathbf{F_{BA}}(I)$ *satisfies* $b \leq g_i^{**} \Leftrightarrow \beta_i^{**} = g_i^{**}$ *and* $b \leq g_i^* \Leftrightarrow \beta_i^{**} = g_i^*$.

This complements the description of $\mathbf{F_{PCS}}(I)$ given in [8].

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